

A generalization of Menon's identity

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Abstract

In this note we give a generalization of the well-known Menon's identity. This is based on applying the Burnside's lemma to a certain group action.

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1 Introduction

One of the most interesting arithmetical identity is due to P.K. Menon [7].

Menon's Identity. *For every positive integer n we have*

$$\sum_{a \in U(\mathbb{Z}_n)} \gcd(n, a-1) = \varphi(n) \tau(n),$$

where $U(\mathbb{Z}_n) = \{a \in \mathbb{Z}_n \mid \gcd(n, a) = 1\}$, φ is the Euler's totient function and $\tau(n)$ is the number of divisors of n .

This identity has many generalizations derived by several authors (see, for example, [1]-[6], [10, 11] and [13]-[20]). An usual technique to prove results of this type is based on the so-called Burnside's lemma (see [8]) concerning group actions.

Burnside's Lemma. *Let G be a finite group acting on a finite set X and let $X^g = \{x \in X \mid g \cdot x = x\}$, for all $g \in G$. Then the number of distinct orbits is*

$$N = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

The starting point for our discussion is given by the open problem in the end of Section 2 of [16] that suggests to apply the Burnside's lemma to the natural action of the group G of upper triangular matrices contained in $GL_r(\mathbb{Z}_n)$, that is

$$G = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ 0 & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{rr} \end{pmatrix} \mid a_{ii} \in U(\mathbb{Z}_n) \forall i=\overline{1, r}, a_{ij} \in \mathbb{Z}_n \forall 1 \leq i < j \leq r \right\}$$

(notice that $|G| = n^{\frac{r(r-1)}{2}} \varphi(n)^r$), on the set

$$X = \mathbb{Z}_n^r = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} \mid x_i \in \mathbb{Z}_n \forall i=\overline{1, r} \right\}.$$

Following this idea we obtained an interesting generalization of the Menon's identity. Denote $\tau_1(n) = \tau(n)$ and $\tau_i(n) = \sum_{d|n} \tau_{i-1}(d)$, for all $i \geq 2$. We

remark that an alternative way to define these functions is $\tau_i = \tau_{i-1} * e$, where $*$ denotes the usual Dirichlet convolution and $e(n) = 1$ for all $n \in \mathbb{N}$. In other words, $\tau_i = e^{(i+1)}$, where $e^{(i+1)}$ is the $(i+1)$ th power under the Dirichlet convolution (see e.g. [9, 12]). Our main result is:

Theorem. *For every positive integers n and r we have*

$$(*) \quad \sum_{\substack{a_{ii} \in U(\mathbb{Z}_n), i=\overline{1, r} \\ a_{ij} \in \mathbb{Z}_n, 1 \leq i < j \leq r}} \prod_{k=1}^r d_k = n^{\frac{r(r-1)}{2}} \varphi(n)^r \tau_r(n),$$

where

$$d_k = \gcd \left(n, \frac{na_{1k}}{\gcd(n, a_{11}-1, a_{12}, \dots, a_{1k-1})}, \frac{na_{2k}}{\gcd(n, a_{22}-1, a_{23}, \dots, a_{2k-1})}, \dots, \frac{na_{k-1k}}{\gcd(n, a_{k-1k-1}-1)}, a_{kk}-1 \right) \forall k = \overline{1, r}.$$

2 Proof of the main theorem

We will proceed by induction on r . Obviously, for $r = 1$ the equality $(*)$ is the Menon's identity.

In the following we will focus on the case $r = 2$ (this is not necessary, but very suggestive for the general implication step). We have to prove that

$$(1) \quad \sum_{\substack{a_{11}, a_{22} \in U(\mathbb{Z}_n) \\ a_{12} \in \mathbb{Z}_n}} \gcd(n, a_{11}-1) \gcd\left(n, \frac{na_{12}}{\gcd(n, a_{11}-1)}, a_{22}-1\right) = n\varphi(n)^2 \tau_2(n).$$

Clearly, two elements $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ of X are contained in the same orbit if and only if there is $g = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in G$ such that $y = g \cdot x$, i.e.

$$(2) \quad \begin{cases} a_{11}x_1 + a_{12}x_2 = y_1 \\ a_{22}x_2 = y_2. \end{cases}$$

We observe that (2) is equivalent to

$$(2') \quad \begin{cases} \langle x_2 \rangle = \langle y_2 \rangle (= H) \\ \langle x_1 H \rangle = \langle y_1 H \rangle \text{ in } \mathbb{Z}_n/H, \end{cases}$$

that means

$$(2'') \quad \begin{cases} o(x_2) = o(y_2) = \delta \in L_n \\ o_H(x_1) = o_H(y_1) = \delta' \in L_{\frac{n}{\delta}}, \end{cases}$$

where for a positive integer m we denote by L_m the lattice of divisors of m . In this way, one obtains

$$(3) \quad N = |\{(\delta, \delta') \mid \delta \in L_n, \delta' \in L_{\frac{n}{\delta}}\}| = \sum_{\delta|n} \tau\left(\frac{n}{\delta}\right) = \sum_{\delta|n} \tau(\delta) = \tau_2(n).$$

Remark. For $r = 2$ an explicit formula of N can be given, namely if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ is the decomposition of n as a product of prime factors, then

$$N = \frac{1}{2^s} \prod_{i=1}^s (\alpha_i + 1)(\alpha_i + 2).$$

Next we observe that for a fixed $g = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in G$ we have $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X^g$ if and only if $g \cdot x = x$, i.e.

$$(4) \quad \begin{cases} (a_{11} - 1)x_1 + a_{12}x_2 = 0 \\ (a_{22} - 1)x_2 = 0. \end{cases}$$

By multiplying the first equation with $\frac{n}{\gcd(n, a_{11}-1)}$, it follows that (4) is equivalent to

$$(4') \quad \begin{cases} \frac{na_{12}}{\gcd(n, a_{11}-1)} x_2 = 0 \\ (a_{22} - 1)x_2 = 0 \end{cases}$$

and consequently to

$$x_2 \in \left\langle \frac{n}{\gcd(n, \frac{na_{12}}{\gcd(n, a_{11}-1)})} \right\rangle \cap \left\langle \frac{n}{\gcd(n, a_{22}-1)} \right\rangle = \left\langle \frac{n}{\gcd(n, \frac{na_{12}}{\gcd(n, a_{11}-1)}, a_{22}-1)} \right\rangle.$$

So, x_2 can be chosen in $\gcd(n, \frac{na_{12}}{\gcd(n, a_{11}-1)}, a_{22}-1)$ ways. Moreover, we easily infer that for each such choice x_1 can be chosen in $\gcd(n, a_{11}-1)$ ways. Hence

$$|X^g| = \gcd(n, a_{11}-1) \gcd\left(n, \frac{na_{12}}{\gcd(n, a_{11}-1)}, a_{22}-1\right),$$

which together with (3) lead to (1), as desired.

Finally, we will prove the general implication step. Assume that $(*)$ holds

for $r-1$. Two elements $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{pmatrix}$ of X belong to the same

orbit if and only if $y = g \cdot x$ for some $g = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ 0 & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{rr} \end{pmatrix} \in G$, i.e.

$$(5) \quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1r}x_r = y_1 \\ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2r}x_r = y_2 \\ \vdots \\ a_{rr}x_r = y_r. \end{cases}$$

The equalities (5) are equivalent to

$$(5') \quad \begin{cases} \langle x_r \rangle = \langle y_r \rangle \\ \langle x_{r-1}H_1 \rangle = \langle y_{r-1}H_1 \rangle \text{ in } \mathbb{Z}_n/H_1 \\ \vdots \\ \langle x_1H_{r-1} \rangle = \langle y_1H_{r-1} \rangle \text{ in } \mathbb{Z}_n/H_{r-1}, \end{cases}$$

where

$$\begin{aligned} H_1 &= \langle x_r \rangle = \langle y_r \rangle, \\ H_2 &= \langle x_{r-1}, x_r \rangle = \langle y_{r-1}, y_r \rangle, \\ &\vdots \\ H_{r-1} &= \langle x_2, x_3, \dots, x_r \rangle = \langle y_2, y_3, \dots, y_r \rangle, \end{aligned}$$

which means

$$(5'') \quad \begin{cases} o(x_r) = o(y_r) = \delta_1 \in L_n \\ o_{H_1}(x_{r-1}) = o_{H_1}(y_{r-1}) = \delta_2 \in L_{\frac{n}{\delta_1}} \\ \vdots \\ o_{H_{r-1}}(x_1) = o_{H_{r-1}}(y_1) = \delta_r \in L_{\frac{n}{\delta_1\delta_2\cdots\delta_{r-1}}} . \end{cases}$$

It is now easy to see that

$$\begin{aligned} (6) \quad N &= | \{ (\delta_1, \delta_2, \dots, \delta_r) \mid \delta_1 \in L_n, \delta_2 \in L_{\frac{n}{\delta_1}}, \dots, \delta_r \in L_{\frac{n}{\delta_1\delta_2\cdots\delta_{r-1}}} \} | = \\ &= \sum_{\delta_1|n} | \{ (\delta_2, \dots, \delta_r) \mid \delta_2 \in L_{\frac{n}{\delta_1}}, \dots, \delta_r \in L_{\frac{n}{\delta_1\delta_2\cdots\delta_{r-1}}} \} | = \cdots = \\ &= \sum_{\delta_1|n} \sum_{\delta_2|\frac{n}{\delta_1}} \cdots \sum_{\delta_{r-1}|\frac{n}{\delta_1\delta_2\cdots\delta_{r-2}}} \tau\left(\frac{n}{\delta_1\delta_2\cdots\delta_{r-1}}\right) = \tau_r(n). \end{aligned}$$

On the other hand, given $g = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ 0 & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{rr} \end{pmatrix} \in G$, we have $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} \in$

X^g if and only if

$$(7) \quad \begin{cases} (a_{11} - 1)x_1 + a_{12}x_2 + \cdots + a_{1r}x_r = 0 \\ (a_{22} - 1)x_2 + a_{23}x_3 + \cdots + a_{2r}x_r = 0 \\ \vdots \\ (a_{rr} - 1)x_r = 0. \end{cases}$$

By multiplying the first equation with $\frac{n}{\gcd(n, a_{11}-1, a_{12}, \dots, a_{1r-1})}$, the second one with $\frac{n}{\gcd(n, a_{22}-1, a_{23}, \dots, a_{2r-1})}$, ..., and the last but one with $\frac{n}{\gcd(n, a_{r-1r-1}-1)}$, (7) becomes a system in x_r that has

$$d_r = \gcd \left(n, \frac{na_{1r}}{\gcd(n, a_{11}-1, a_{12}, \dots, a_{1r-1})}, \dots, \frac{na_{r-1r}}{\gcd(n, a_{r-1r-1}-1)}, a_{rr}-1 \right)$$

solutions, namely $x_r \in \langle \frac{n}{d_r} \rangle$. Put $x_r = \gamma \frac{n}{d_r}$ with $\gamma \in \{0, 1, \dots, d_r - 1\}$. Then (7) can be rewritten as

$$(7') \quad \begin{cases} (a_{11} - 1)x_1 + a_{12}x_2 + \dots + a_{1r-1}x_{r-1} = -\gamma \frac{n}{d_r} a_{1r} \\ (a_{22} - 1)x_2 + a_{23}x_3 + \dots + a_{2r-1}x_{r-1} = -\gamma \frac{n}{d_r} a_{2r} \\ \vdots \\ (a_{r-1r-1} - 1)x_{r-1} = -\gamma \frac{n}{d_r} a_{r-1r}. \end{cases}$$

If $(x_1^0, x_2^0, \dots, x_{r-1}^0)$ is a particular solution of (7'), then one obtains a homogeneous system

$$(7'') \quad \begin{cases} (a_{11} - 1)(x_1 - x_1^0) + a_{12}(x_2 - x_2^0) + \dots + a_{1r-1}(x_{r-1} - x_{r-1}^0) = 0 \\ (a_{22} - 1)(x_2 - x_2^0) + a_{23}(x_3 - x_3^0) + \dots + a_{2r-1}(x_{r-1} - x_{r-1}^0) = 0 \\ \vdots \\ (a_{r-1r-1} - 1)(x_{r-1} - x_{r-1}^0) = 0 \end{cases}$$

with $\prod_{k=1}^{r-1} d_k$ solutions by the inductive hypothesis. We infer that

$$|X^g| = \prod_{k=1}^r d_k,$$

which together with (6) lead to the equality (*). This completes the proof. ■

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